Int. J. Solids Structures Vol. 30, No. 23, pp. 3211–3231, 1993
Printed in Great Britain

0020–7683/93 $6.00+ .00

AN ANALYTIC MODEL FOR THERMOELASTIC PROPERTIES OF COMPOSITE LAMINATES CONTAINING TRANSVERSE MATRIX CRACKS

PETER GUDMUNDSON† and WEILIN ZANG
Swedish Institute of Composites, Box 271, S-941 26 Piteå, Sweden

(*Received* 1 *February* 1993; *in revised form* 18 *May* 1993)

Abstract—An analytical model for the prediction of the thermoelastic properties of composite laminates containing matrix cracks is presented. In particular, transverse matrix cracks with their crack surfaces parallel to the fibre direction and perpendicular to the laminate plane are treated. Two- and three-dimensional laminates of arbitrary layup configurations are covered by the model. The presented expressions for stiffnesses, thermal expansion coefficients, strain contributions from release of residual stresses and local average ply stresses and strains do solely contain known ply property data and matrix crack densities. The key to the model is the judicious use of a known analytical solution for a row of cracks in an infinite isotropic medium. The model has been verified against numerically determined stiffnesses, thermal expansion coefficients and local average ply stresses for matrix cracked angle-ply and cross-ply laminates. Comparisons to experimental data for cross-ply laminates are also presented. It is shown that the present model to a very good accuracy can predict thermoelastic properties of matrix cracked composite laminates at varying matrix crack densities and layup configurations.

1. INTRODUCTION

When composite laminates are mechanically loaded, different kinds of damage modes such as transverse matrix cracking, delaminations and fibre fractures develop before final failure of the laminate. Transverse matrix cracking is often the first observed damage mode. This mode is generally not critical from a final fracture point of view. The matrix cracks can however initiate more severe damage such as delaminations and fibre fractures. A consequence of matrix cracking is that both local and global stress and strain redistributions occur in a laminate. For example, local stress concentrations close to the tips of the matrix cracks may cause the initiation of local delaminations and/or fibre fractures. Since matrix cracks generally are initiated long before final fracture of a structure, they should be taken into account in the design in order to fully utilize the load bearing capacity of a composite structure.

In order to simulate the mechanical behaviour of a matrix cracking composite laminate, the constitutive law which defines the stress–strain relationship for the laminate must include the effects of transverse matrix cracks. Compared to linear, elastic laminate theory the constitutive law should basically be extended to include two main aspects. First of all, criteria for transverse matrix crack initiation and growth must be implemented. Secondly, at given matrix crack densities the model must enable reliable estimations of the relationship between global stresses and strains as well as means for the estimation of local ply stresses and strains. In this paper this second aspect is addressed.

The simplest way to model transverse matrix cracks in composite laminates is to completely neglect the transverse stiffnesses of cracked plies. This method is generally called the ply discount method. The ply discount method will underestimate the stiffnesses of cracked laminates, since cracked plies can take some loading. Therefore the gradual changes of laminate properties with increasing matrix crack densities can never be covered by the ply discount method.

A relatively simple way to include the effects of load transfer between micro cracked plies and their neighbours is to apply a so-called shear lag analysis. In this theory, the load transfer between plies is assumed to take place in shear layers between neighbouring plies.

†Present address: Department of Solid Mechanics, Royal Institute of Technology, S-100 44 Stockholm, Sweden.

The thicknesses and stiffnesses of these shear layers are generally unknown. The variations in the thickness direction of local ply stresses and strains are also neglected in the shear lag theory. Another aspect of the shear lag theory is that it is not obvious how it should be applied for layup configurations other than cross plies. The shear lag theory has however successfully been applied to cross-ply laminates (Highsmith and Reifsnider, 1982; Lim and Hong, 1989; Han and Hahn, 1989; Tan and Nuismer, 1989).

By application of the principle of minimum complementary potential energy Hashin (1985, 1987, 1988) derived estimates for stiffnesses, thermal expansion coefficients as well as local ply stresses of matrix cracked cross-ply laminates. He showed that the estimates were in good agreement with experimental data. An advantage with Hashin's method is that strict lower bounds for stiffnesses are obtained. Varna and Berglund (1991) have later by use of more extensive trial stress functions made some improvements to the Hashin model. A disadvantage of the Hashin method is that it is extremely difficult to use for laminate layups other than cross plies. To the authors' knowledge the method has only been applied to cross plies.

Laws *et al.* (1983) and Dvorak *et al.* (1985) have estimated stiffnesses and thermal expansion coefficients of matrix cracked composite plies by use of self consistent approximations. The self consistent stiffnesses were derived for an infinite, homogeneous, matrix cracked material. Laminate stiffnesses can then be derived by use of laminate theory and appropriate self consistent ply stiffnesses.

An alternative way to describe the mechanical behaviour of matrix cracked laminates is to apply concepts of damage mechanics. Talreja (1985, 1986) and Allen *et al.* (1987a,b) have derived models for laminate stiffnesses in terms of internal damage state parameters. In order to apply the models, it is necessary to fit certain parameters to experimental or numerical data. For a matrix cracked cross-ply laminate Lee and Allen (1989) and Allen and Lee (1991) have derived approximate relations between the internal damage state parameter and laminate stiffnesses. They determined approximate solutions for local stresses and strains by use of the principle of minimum potential energy. In this way upper bounds for laminate stiffnesses could be derived.

Gudmundson and Östlund (1992a, b, c) and Gudmundson *et al.* (1992) have shown that for dilute and infinite matrix crack densities respectively asymptotic expressions of high accuracy for the laminate stiffnesses can be derived in closed form for laminates of arbitrary layups. Asymptotic expressions for thermal expansion coefficients, strain contributions from release of residual stresses as well as average local stresses and strains were also determined. The dilute formulation is principally based on knowledge of the average crack opening displacement of a single matrix crack in the laminate as a function of the applied loading. Gudmundson and Östlund (1992a) showed that this average crack opening displacement was to a very good approximation given by an equivalent crack in an infinite, transversely isotropic medium. By use of this approximate expression for average crack opening displacements, closed form expressions could be derived for laminate stiffnesses, thermal expansion coefficients, strain contributions from release of residual stresses as well as average local ply stresses and strains (Gudmundson and Östlund, 1992a, b, c; Gudmundson *et al.*, 1992). There was no restriction concerning laminate layup or whether internal or edge micro cracks were considered. The theory was formulated for a general three-dimensional laminate. Comparisons to numerically and experimentally determined laminate stiffnesses, thermal expansion coefficients and local stresses and strains for laminates of different layups proved that the dilute theory worked extremely well up to certain matrix crack densities and that estimates based on infinite crack densities were good for matrix crack densities above certain limits (Gudmundson and Östlund, 1992a, b, c). For intermediate crack densities the differences between theory and numerically or experimentally determined data were most significant. Intermediate crack densities are here considered to be around one crack per unit thickness of a cracked ply. The reason for the discrepancies at intermediate crack densities is that interactions between neighbouring cracks become of importance. This effect is not taken into account in the dilute theory. In addition, intermediately cracked plies do still carry some load transverse to the cracks and this effect is neglected in the theoretical estimate for infinite crack densities.

Experimental observations (Highsmith and Reifsnider, 1982) have shown that the matrix crack density generally reaches a saturation state which can be characterized as an intermediate crack density. It would therefore be of advantage if the dilute and infinite theory developed by Gudmundson and Östlund (1992a, b, c) and Gudmundson *et al.* (1992) could be improved in the range of intermediate crack densities, but still keeping the nice features such as closed form expressions and applicability to laminates of arbitrary layups. In the present paper a significant improvement to the previous theory will be presented. It will be shown that the modified theory coincides with the dilute theory at small crack densities and with the infinite theory at large crack densities. At intermediate crack densities, it will be proved that the present theory is in very good agreement with numerically obtained data for angle-ply and cross-ply laminates. The key to the present theory is the judicious use of an existing analytical solution for a row of cracks in an infinite, homogeneous, isotropic medium (Benthem and Koiter, 1972; Tada *et al.*, 1973). Laminate stiffnesses, thermal expansion coefficients, strain contributions from release of residual stresses, average local stresses and strains will be expressed in closed forms only involving algebraic manipulations of known quantities such as ply stiffnesses, ply thermal expansion coefficients and micro crack densities. Both internal and edge cracks can be considered. The theory is developed for a three-dimensional laminate. As a special case the expressions for a two-dimensional laminate are derived.

2. THEORETICAL BASIS

2.1. *Three-dimensional laminate theory without transverse matrix cracks*

The stiffness and compliance tensors of a three-dimensional laminate without transverse matrix cracks have previously been derived by other researchers [see for example Pagano (1974) and Sun and Li (1988)]. However, in order to make the subsequent theoretical developments in Sections 2.2–2.3 easier to follow, three-dimensional laminate theory in a compact notation will here briefly be summarized. Laminate theory is actually a homogenization process. Instead of using the properties of each ply, a set of effective properties are employed and the laminate structure is treated as if it were made of an equivalent homogeneous material. It should be stressed that the homogenized equations (laminate theory) do have certain limitations. The existence of boundary layer effects (for composite laminates often called edge effects) cannot be modelled. The theory also breaks down when characteristic length scales of homogenized deformation variations are of the same order as microstructural dimensions (ply thicknesses for composite laminates). There are two basic tasks of a laminate theory, (1) to establish the relations between ply material properties (such as compliances and thermal expansion coefficients) and the effective properties, (2) to recover the ply stresses and strains from known effective stresses and strains.

A general three-dimensional thick laminate without matrix cracks is considered here. The laminate consists of N different types of plies. A type of ply is defined by ply material properties, layup angle and thickness. For laminates without matrix cracks, the global average stresses $\bar{\boldsymbol{\sigma}}$ and strains $\bar{\boldsymbol{\varepsilon}}$ are defined as:

$$\left.\begin{aligned} \bar{\boldsymbol{\sigma}} &= \sum_{k=1}^{N} v^k \boldsymbol{\sigma}^k \\ \bar{\boldsymbol{\varepsilon}} &= \sum_{k=1}^{N} v^k \boldsymbol{\varepsilon}^k \\ \sum_{k=1}^{N} v^k &= 1 \end{aligned}\right\}, \tag{1}$$

where $\boldsymbol{\sigma}^k$ denotes the ply average stresses, $\boldsymbol{\varepsilon}^k$ the ply average strains and v^k the volume fraction of ply k. Throughout this paper, variables with superscript bars denote the global properties and variables with superscript letters denote ply properties. For uncracked laminates under homogeneous deformation states, the ply stresses and strains are constant within each ply. In this case, there is no difference between ply averages and local values of stresses and strains.

Two sets of coordinate systems will be employed in the present study. One is the global coordinate system with its axes X_1 and X_2 lying in the same plane as the plies and the axis X_3 perpendicular to the plane of plies. The other coordinate system is the local coordinate system for each ply with its axis Y_1 along the fibre direction, axis Y_2 perpendicular to the fibre direction but in the ply plane and axis Y_3 parallel to the axis X_3. In the following, the stresses, strains and thermal expansion coefficients will be partitioned into in-plane parts and out-of plane parts:

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{\sigma}_{\mathrm{I}} \\ \boldsymbol{\sigma}_{\mathrm{O}} \end{pmatrix}, \quad \boldsymbol{\varepsilon} = \begin{pmatrix} \boldsymbol{\varepsilon}_{\mathrm{I}} \\ \boldsymbol{\varepsilon}_{\mathrm{O}} \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\alpha}_{\mathrm{I}} \\ \boldsymbol{\alpha}_{\mathrm{O}} \end{pmatrix}, \tag{2}$$

where

$$\boldsymbol{\sigma}_{\mathrm{I}} = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}, \quad \boldsymbol{\varepsilon}_{\mathrm{I}} = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix}, \quad \boldsymbol{\alpha}_{\mathrm{I}} = \begin{pmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{pmatrix} \tag{3}$$

are in-plane stresses, strains and thermal expansion coefficients and

$$\boldsymbol{\sigma}_{\mathrm{O}} = \begin{pmatrix} \sigma_{33} \\ \sigma_{13} \\ \sigma_{23} \end{pmatrix}, \quad \boldsymbol{\varepsilon}_{\mathrm{O}} = \begin{pmatrix} \varepsilon_{33} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{pmatrix}, \quad \boldsymbol{\alpha}_{\mathrm{O}} = \begin{pmatrix} \alpha_{33} \\ 2\alpha_{13} \\ 2\alpha_{23} \end{pmatrix}, \tag{4}$$

are out-of-plane stresses, strains and termal expansion coefficients. Using these notations, the relationship between global effective stresses and strains reads:

$$\begin{aligned} \bar{\boldsymbol{\varepsilon}} = \begin{pmatrix} \bar{\boldsymbol{\varepsilon}}_{\mathrm{I}} \\ \bar{\boldsymbol{\varepsilon}}_{\mathrm{O}} \end{pmatrix} &= \bar{\mathbf{S}}\bar{\boldsymbol{\sigma}} + \bar{\boldsymbol{\alpha}}\Delta T \\ &= \begin{pmatrix} \bar{\mathbf{S}}_{\mathrm{II}} & \bar{\mathbf{S}}_{\mathrm{IO}} \\ (\bar{\mathbf{S}}_{\mathrm{IO}})^{\mathrm{T}} & \bar{\mathbf{S}}_{\mathrm{OO}} \end{pmatrix} \begin{pmatrix} \bar{\boldsymbol{\sigma}}_{\mathrm{I}} \\ \bar{\boldsymbol{\sigma}}_{\mathrm{O}} \end{pmatrix} + \begin{pmatrix} \bar{\boldsymbol{\alpha}}_{\mathrm{I}} \\ \bar{\boldsymbol{\alpha}}_{\mathrm{O}} \end{pmatrix} \Delta T. \end{aligned} \tag{5}$$

Similarly, the relation between the ply stresses and strains can be written as

$$\begin{aligned} \boldsymbol{\varepsilon}^k = \begin{pmatrix} \boldsymbol{\varepsilon}_{\mathrm{I}}^k \\ \boldsymbol{\varepsilon}_{\mathrm{O}}^k \end{pmatrix} &= \mathbf{S}^k(\boldsymbol{\sigma}^k - \boldsymbol{\sigma}^{k(r)}) + \boldsymbol{\alpha}^k \Delta T \\ &= \begin{pmatrix} \mathbf{S}_{\mathrm{II}}^k & \mathbf{S}_{\mathrm{IO}}^k \\ (\mathbf{S}_{\mathrm{IO}}^k)^{\mathrm{T}} & \mathbf{S}_{\mathrm{OO}}^k \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma}_{\mathrm{I}}^k - \boldsymbol{\sigma}_{\mathrm{I}}^{k(r)} \\ \boldsymbol{\sigma}_{\mathrm{O}}^k \end{pmatrix} + \begin{pmatrix} \boldsymbol{\alpha}_{\mathrm{I}}^k \\ \boldsymbol{\alpha}_{\mathrm{O}}^k \end{pmatrix} \Delta T, \end{aligned} \tag{6}$$

where $\boldsymbol{\sigma}^{k(r)}$ denote residual stresses which may be present due to other reasons than thermal mismatch, for example chemical shrinkage during the manufacturing process. In eqns (5), (6), $\bar{\mathbf{S}}$ and $\mathbf{S}^k$ are the 6×6 effective and ply compliances respectively and the superscript T indicates the transpose of a matrix. The compliances in eqns (5), (6) have been divided into 3×3 sub-matrices, $\bar{\mathbf{S}}_t$ and $\mathbf{S}_t^k$ ($t = \mathrm{II}, \mathrm{IO}, \mathrm{OO}$). The effective compliance tensor $\bar{\mathbf{S}}$ and the effective thermal expansion tensor $\bar{\boldsymbol{\alpha}}$ are still to be defined. It should be pointed out that due to equilibrium, the out-of-plane residual stresses and the volume average of in-plane residual stresses do vanish.

From the compatibility and equilibrium conditions in the laminate, the following relations result:

$$\boldsymbol{\varepsilon}_{\mathrm{I}}^k = \bar{\boldsymbol{\varepsilon}}_{\mathrm{I}}, \quad \boldsymbol{\sigma}_{\mathrm{O}}^k = \bar{\boldsymbol{\sigma}}_{\mathrm{O}}. \tag{7}$$

Thus from eqns (6), (7)

$$\boldsymbol{\sigma}_{\mathrm{I}}^{k} = (\mathbf{S}_{\mathrm{II}}^{k})^{-1}(\bar{\boldsymbol{\varepsilon}}_{\mathrm{I}} - \mathbf{S}_{\mathrm{IO}}^{k}\bar{\boldsymbol{\sigma}}_{\mathrm{O}} - \boldsymbol{\alpha}_{\mathrm{I}}^{k}\Delta T) + \boldsymbol{\sigma}_{\mathrm{I}}^{k(r)}. \tag{8}$$

An application of eqn (1) and a rearrangement of the resulting equation yield

$$\bar{\boldsymbol{\varepsilon}}_{\mathrm{I}} = \bar{\mathbf{S}}_{\mathrm{II}}\bar{\boldsymbol{\sigma}}_{\mathrm{I}} + \bar{\mathbf{S}}_{\mathrm{IO}}\bar{\boldsymbol{\sigma}}_{\mathrm{O}} + \bar{\boldsymbol{\alpha}}_{\mathrm{I}}\Delta T, \tag{9}$$

where

$$\left.\begin{aligned} \bar{\mathbf{S}}_{\mathrm{II}} &= \left[\sum_{k=1}^{N} v^{k}(\mathbf{S}_{\mathrm{II}}^{k})^{-1}\right]^{-1} \\ \bar{\mathbf{S}}_{\mathrm{IO}} &= \bar{\mathbf{S}}_{\mathrm{II}}\left[\sum_{k=1}^{N} v^{k}(\mathbf{S}_{\mathrm{II}}^{k})^{-1}\mathbf{S}_{\mathrm{IO}}^{k}\right] \\ \bar{\boldsymbol{\alpha}}_{\mathrm{I}} &= \bar{\mathbf{S}}_{\mathrm{II}}\left[\sum_{k=1}^{N} v^{k}(\mathbf{S}_{\mathrm{II}}^{k})^{-1}\boldsymbol{\alpha}_{\mathrm{I}}^{k}\right] \end{aligned}\right\}. \tag{10}$$

In eqn (9), the fact that the volume average of residual stresses vanishes has been utilized. Similarly, the following equation results from eqns (6), (7):

$$\boldsymbol{\varepsilon}_{\mathrm{O}}^{k} = (\mathbf{S}_{\mathrm{IO}}^{k})^{\mathrm{T}}(\boldsymbol{\sigma}_{\mathrm{I}}^{k} - \boldsymbol{\sigma}_{\mathrm{I}}^{k(r)}) + \mathbf{S}_{\mathrm{OO}}^{k}\bar{\boldsymbol{\sigma}}_{\mathrm{O}} + \boldsymbol{\alpha}_{\mathrm{O}}^{k}\Delta T. \tag{11}$$

An application of eqn (1) yields

$$\bar{\boldsymbol{\varepsilon}}_{\mathrm{O}} = (\bar{\mathbf{S}}_{\mathrm{IO}})^{\mathrm{T}}\bar{\boldsymbol{\sigma}}_{\mathrm{I}} + \bar{\mathbf{S}}_{\mathrm{OO}}\bar{\boldsymbol{\sigma}}_{\mathrm{O}} + \bar{\boldsymbol{\alpha}}_{\mathrm{O}}\Delta T, \tag{12}$$

where

$$\left.\begin{aligned} \bar{\mathbf{S}}_{\mathrm{OO}} &= (\bar{\mathbf{S}}_{\mathrm{IO}})^{\mathrm{T}}(\bar{\mathbf{S}}_{\mathrm{II}})^{-1}\bar{\mathbf{S}}_{\mathrm{IO}} + \sum_{k=1}^{N} v^{k}[\mathbf{S}_{\mathrm{OO}}^{k} - (\mathbf{S}_{\mathrm{IO}}^{k})^{\mathrm{T}}(\mathbf{S}_{\mathrm{II}}^{k})^{-1}\mathbf{S}_{\mathrm{IO}}^{k}] \\ \bar{\boldsymbol{\alpha}}_{\mathrm{O}} &= (\bar{\mathbf{S}}_{\mathrm{IO}})^{\mathrm{T}}(\bar{\mathbf{S}}_{\mathrm{II}})^{-1}\bar{\boldsymbol{\alpha}}_{\mathrm{I}} + \sum_{k=1}^{N} v^{k}[\boldsymbol{\alpha}_{\mathrm{O}}^{k} - (\mathbf{S}_{\mathrm{IO}}^{k})^{\mathrm{T}}(\mathbf{S}_{\mathrm{II}}^{k})^{-1}\boldsymbol{\alpha}_{\mathrm{I}}^{k}] \end{aligned}\right\}. \tag{13}$$

In summary, eqns (10), (13) establish the relations between the local material properties and the effective laminate properties. The ply stresses and strains can be recovered from eqns (7), (8), (11).

It should be pointed out that the equations derived above yield as a special case the effective thermoelastic parameters given by the standard two-dimensional laminate theory. In this case, $\bar{\boldsymbol{\sigma}}_{\mathrm{O}} = \mathbf{0}$ and usually only the in-plane properties are considered.

2.2. *Thermoelastic properties for composite laminates containing transverse matrix cracks*

A general three-dimensional thick composite laminate containing transverse matrix cracks is considered (see Fig. 1). The ply material properties and the number of transverse

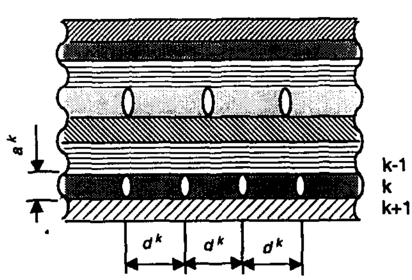

Fig. 1. A general three-dimensional laminate with micro cracks in ply k.

matrix cracks in each ply are assumed to be known. The matrix crack density in a typical ply k is denoted as ρ^k and is in this paper defined as the ratio between ply thickness and average distance between micro cracks

$$\rho^k = a^k/d_k. \tag{14}$$

The parameters in eqn (14) are defined in Fig. 1 in which a case of equidistant cracks is presented.

In the homogenization procedure for cracked laminates discussed below, a representative volume V [cf. Hill (1963)] which is large in comparison with ply thicknesses and distances between matrix cracks is implicitly considered. On the outer boundary Γ^{out} of V, displacements or tractions which are consistent with a homogeneous deformation field are prescribed. The displacement or traction boundary conditions will induce a surface layer effect, but within V a macroscopically homogeneous state will develop. Concerning volume averages, the effects of the surface layer can be made negligible by taking V large enough. In the analysis below, the various considered averages and effective properties should be interpreted in this sense.

The terms effective and average strains are often interchangeably referring to the same properties. When matrix cracks occur, however, these terms do have different meanings. In the present paper, effective strains are the strains which would be measured on a global scale and the average strains are the averages of strains experienced by the material in different plies. The difference between effective and average strains is due to the contribution from crack opening displacements. The global effective strains are defined as [cf. Hill (1963)]

$$\bar{\varepsilon}_{ij}^{(e)} = \frac{1}{2V}\int_{\Gamma^{\text{out}}} (u_i n_j + u_j n_i)\, \mathrm{d}\Gamma, \tag{15}$$

where u_i denotes the displacements, n_i the unit normal vector on Γ^{out} (the outer boundary surface of a volume V which is large in comparison to distances between cracks and ply thicknesses) and the superscript (e) the effective variables. For stresses, there is no distinction between global effective and average stresses. This follows immediately from the relation between global effective and average stresses given by Hill (1963) [see also the review by Kachanov (1992)]. The global average stresses are defined as

$$\bar{\sigma}_{ij}^{(a)} = \sum_{k=1}^{N} v^k \sigma_{ij}^{k(a)}, \tag{16}$$

where the superscript (a) indicates average variables.

The ply effective strains can in the same way be defined as

$$\varepsilon_{ij}^{k(e)} = \frac{1}{2V^k}\int_{\Gamma^{k\text{out}}} (u_i^k n_j^k + u_j^k n_i^k)\, \mathrm{d}\Gamma, \tag{17}$$

where V^k is the volume of ply k within V and the integral is only performed on the outer boundary surfaces of ply k. It is obvious that

$$\frac{1}{2V}\int_{\Gamma^{\text{out}}} (u_i n_j + u_j n_i)\, \mathrm{d}\Gamma = \sum_{k=1}^{N} v^k \left[\frac{1}{2V^k}\int_{\Gamma^{k\text{out}}} (u_i^k n_j^k + u_j^k n_i^k)\, \mathrm{d}\Gamma\right]. \tag{18}$$

Equations (15), (17), (18) thus imply that

$$\bar{\varepsilon}_{ij}^{(e)} = \sum_{k=1}^{N} v^k \varepsilon_{ij}^{k(e)}. \tag{19}$$

Equations (16), (19) establish two fundamental relations for a laminate theory taking matrix cracks into account.

By an application of the divergence theorem, the integral for the effective ply strains can be divided into two parts as

$$\varepsilon_{ij}^{k(e)} = \frac{1}{2V^k}\int_{V^k}(u_{i,j}^k + u_{j,i}^k)\,\mathrm{d}V_k - \frac{1}{2V^k}\int_{\Gamma^{kc}}(u_i^k n_j^k + u_j^k n_i^k)\,\mathrm{d}\Gamma, \tag{20}$$

where Γ^{kc} denotes the matrix crack surfaces in ply k and the positive normal directions are defined in Fig. 2. The first integral in eqn (20) is the ply average strain:

$$\varepsilon_{ij}^{k(a)} = \frac{1}{2V^k}\int_{V^k}(u_{j,i}^k + u_{i,j}^k)\,\mathrm{d}\Gamma. \tag{21}$$

Since the normal vector on crack surfaces is constant, the second integral in eqn (20) can be evaluated as

$$\begin{aligned}\Delta\varepsilon_{ij}^k &= \frac{-1}{2V^k}\int_{\Gamma^{kc}}(u_i^k n_j^k + u_j^k n_i^k)\,\mathrm{d}\Gamma \\ &= \frac{\rho^k}{2a^k}(\Delta\bar{u}_i^k n_j^{k(-)} + \Delta\bar{u}_j^k n_i^{k(-)}),\end{aligned} \tag{22}$$

where

$$\Delta\bar{u}_i^k = \frac{1}{a^k}\int_0^{a^k}(u_i^{k(+)} - u_i^{k(-)})\,\mathrm{d}a^k = \frac{1}{a^k}\int_0^{a^k}\Delta u_i^k\,\mathrm{d}a^k \tag{23}$$

is the average crack opening displacement for ply k and $\Delta\varepsilon_{ij}^k$ the average incremental strains due to crack opening displacements. In eqns (22), (23), the superscripts (+) and (−) denote the upper and lower crack surfaces respectively. The ply effective strains can thus be expressed as

$$\varepsilon_{ij}^{k(e)} = \varepsilon_{ij}^{k(a)} + \Delta\varepsilon_{ij}^k, \tag{24}$$

where the average ply strain $\varepsilon_{ij}^{k(a)}$ is given by eqn (21) and the strain increment due to matrix cracks $\Delta\varepsilon_{ij}^k$ by eqn (22). Expressions like eqn (24) have previously been derived and applied

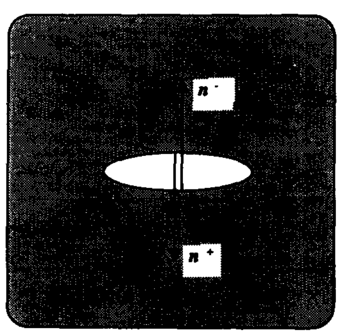

Fig. 2. Definition of the normal vectors on crack surfaces.

by other researchers, see for example the review on the effective elastic properties of cracked solids by Kachanov (1992).

The task is now to derive expressions for effective global strains, effective and average ply strains as well as average global and ply stresses for the laminate under the action of certain loading systems. From this information, effective global properties, e.g. effective compliances, effective thermal expansion coefficients and the contribution to the effective global strains from release of residual stresses, can be obtained. For this purpose, the laminate structure with matrix cracks is subjected to prescribed effective in-plane strains $\bar{\varepsilon}_{\mathrm{I}}^{*}$ and out-of-plane stresses $\bar{\sigma}_{\mathrm{O}}^{*}$ as well as a temperature change ΔT^{*}. In addition, a residual stress state $\sigma_{\mathrm{I}}^{k(\mathrm{r})}$ $(k = 1, 2, \ldots, N)$ is assumed to exist prior to micro cracking. Readers may ask why such a particular loading system ($\bar{\varepsilon}_{\mathrm{I}}^{*}$ and $\bar{\sigma}_{\mathrm{O}}^{*}$) has been chosen. There are basically two reasons. First, from compatibility and equilibrium the ply effective in-plane strains and ply average out-of-plane stresses are immediately defined. This simplifies the required algebraic manipulations in determination of thermoelastic properties. Secondly and most importantly, accurate estimations of average crack opening displacements can be derived for this particular loading system ($\bar{\varepsilon}_{\mathrm{I}}^{*}$, $\bar{\sigma}_{\mathrm{O}}^{*}$, ΔT^{*} and $\sigma_{\mathrm{I}}^{k(\mathrm{r})}$). This will be further discussed in Section 2.3.

Since only linear elasticity is considered, the problem can be solved by a superposition of two problems. In the first problem, the laminate without matrix cracks loaded by prescribed $\bar{\varepsilon}_{\mathrm{I}}^{*}$, $\bar{\sigma}_{\mathrm{O}}^{*}$, ΔT^{*} and $\sigma_{\mathrm{I}}^{k(\mathrm{r})}$ $(k = 1, 2, \ldots, N)$ is considered. This problem can be solved by application of the ordinary laminate theory (see Section 2.1). In particular, the ply stresses can be expressed in terms of the prescribed loading. The tractions on prospective crack surfaces can be written as

$$\boldsymbol{\tau}^{k} = \mathbf{A}^{k}\bar{\boldsymbol{\varepsilon}}_{\mathrm{I}}^{*} + \mathbf{B}^{k}\bar{\boldsymbol{\sigma}}_{\mathrm{O}}^{*} + (\mathbf{C}^{k} - \mathbf{A}^{k}\bar{\boldsymbol{\alpha}}_{\mathrm{I}})\Delta\mathbf{T}^{*} + \boldsymbol{\tau}^{k(\mathrm{r})}, \tag{25}$$

where

$$\left.\begin{aligned} \mathbf{A}^{k} &= \mathbf{N}_{\mathrm{I}}^{k}(\mathbf{S}_{\mathrm{II}}^{k})^{-1} \\ \mathbf{B}^{k} &= -\mathbf{N}_{\mathrm{I}}^{k}(\mathbf{S}_{\mathrm{II}}^{k})^{-1}\mathbf{S}_{\mathrm{IO}}^{k} + \mathbf{N}_{\mathrm{O}}^{k} \\ \mathbf{C}^{k} &= \mathbf{A}^{k}(\bar{\boldsymbol{\alpha}}_{\mathrm{I}} - \boldsymbol{\alpha}_{\mathrm{I}}^{k}) \\ \boldsymbol{\tau}^{k(\mathrm{r})} &= \mathbf{N}_{\mathrm{I}}^{k}\boldsymbol{\sigma}_{\mathrm{I}}^{k(\mathrm{r})} \end{aligned}\right\}, \tag{26}$$

and $\bar{\boldsymbol{\alpha}}_{\mathrm{I}}$ is given in eqn (10). The matrices $\mathbf{N}_{\mathrm{I}}^{k}$ and $\mathbf{N}_{\mathrm{O}}^{k}$ in eqn (26) represent the unit normal vector n_{j}^{k} on the crack surfaces in ply k [the superscript $(-)$ according to eqn (22) is here omitted]:

$$\mathbf{N}_{\mathrm{I}}^{k} = \begin{pmatrix} n_{1}^{k} & 0 & n_{2}^{k} \\ 0 & n_{2}^{k} & n_{1}^{k} \\ 0 & 0 & 0 \end{pmatrix}, \tag{27}$$

$$\mathbf{N}_{\mathrm{O}}^{k} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & n_{1}^{k} & n_{2}^{k} \end{pmatrix}. \tag{28}$$

According to the definition of the coordinate systems, the normal vector on crack surfaces always lies in the local 1, 2-plane, i.e. $n_{3}^{k} = 0$.

In the second problem, the tractions on crack surfaces resulting from the first problem [eqn (25)] are released under vanishing effective global in-plane strains ($\bar{\boldsymbol{\varepsilon}}_{\mathrm{I}}^{(\mathrm{e})} = \mathbf{0}$) and average global out-of-plane stresses ($\bar{\boldsymbol{\sigma}}_{\mathrm{O}}^{(\mathrm{a})} = \mathbf{0}$). The solution to this problem will enable the determination of average crack opening displacements. The average crack opening displacements in a typical ply k will in a general case linearly depend on all crack surface tractions. Thus

$$\Delta\bar{\mathbf{u}}^k = a^k \sum_{i=1}^{N} \boldsymbol{\beta}^{ki}\boldsymbol{\tau}^i, \tag{29}$$

where $\boldsymbol{\beta}^{ki}$ $(k, i = 1, 2, \ldots, N)$ are 3×3 matrices which depend on laminate layup, ply properties and matrix crack densities. The determination of these matrices will be discussed in Section 2.3. The average increment strains due to crack opening displacements for ply k can, according to eqns (22), (25)–(29), be written as

$$\begin{aligned}\Delta\boldsymbol{\varepsilon}_{\mathrm{I}}^k &= \rho^k/a^k(\mathbf{N}_{\mathrm{I}}^k)^{\mathrm{T}}\Delta\bar{\mathbf{u}}^k \\ &= \rho^k(\mathbf{N}_{\mathrm{I}}^k)^{\mathrm{T}}\sum_{i=1}^{N}\boldsymbol{\beta}^{ki}[\mathbf{A}^i\bar{\boldsymbol{\varepsilon}}_{\mathrm{I}}^* + \mathbf{B}^i\bar{\boldsymbol{\sigma}}_{\mathrm{O}}^* + (\mathbf{C}^i - \mathbf{A}^i\bar{\boldsymbol{\alpha}}_{\mathrm{I}})\Delta\mathbf{T}^* + \boldsymbol{\tau}^{i(r)}],\end{aligned} \tag{30a}$$

$$\begin{aligned}\Delta\boldsymbol{\varepsilon}_{\mathrm{O}}^k &= \rho^k/a^k(\mathbf{N}_{\mathrm{O}}^k)^{\mathrm{T}}\Delta\bar{\mathbf{u}}^k \\ &= \rho^k(\mathbf{N}_{\mathrm{O}}^k)^{\mathrm{T}}\sum_{k=1}^{N}\boldsymbol{\beta}^{ki}[\mathbf{A}^i\bar{\boldsymbol{\varepsilon}}_{\mathrm{I}}^* + \mathbf{B}^i\bar{\boldsymbol{\sigma}}_{\mathrm{O}}^* + (\mathbf{C}^i - \mathbf{A}^i\bar{\boldsymbol{\alpha}}_{\mathrm{I}})\Delta\mathbf{T}^* + \boldsymbol{\tau}^{i(r)}].\end{aligned} \tag{30b}$$

The effective ply strains in ply k (problem 1 + problem 2) are given by

$$\boldsymbol{\varepsilon}_{(\mathrm{c})}^{k(\mathrm{e})} = \boldsymbol{\varepsilon}_{(\mathrm{c})}^{k(\mathrm{a})} + \Delta\boldsymbol{\varepsilon}^k, \tag{31}$$

where the subscript (c) denotes the variables for the cracked laminate (problem 1 + problem 2) and $\Delta\boldsymbol{\varepsilon}^k$ is given in eqn (30). In addition, the relations between stresses and strains for a laminate with matrix cracks can be expressed as

$$\left.\begin{aligned}\bar{\boldsymbol{\varepsilon}}_{(\mathrm{c})}^{(\mathrm{e})} &= \bar{\mathbf{S}}_{(\mathrm{c})}\bar{\boldsymbol{\sigma}}_{(\mathrm{c})}^{(\mathrm{a})} + \bar{\boldsymbol{\alpha}}_{(\mathrm{c})}\Delta T + \bar{\boldsymbol{\varepsilon}}_{(\mathrm{c})}^{(\mathrm{r})} \\ \boldsymbol{\varepsilon}_{(\mathrm{c})}^{k(\mathrm{a})} &= \mathbf{S}^k(\boldsymbol{\sigma}_{(\mathrm{c})}^{k(\mathrm{a})} - \boldsymbol{\sigma}^{k(\mathrm{r})}) + \boldsymbol{\alpha}^k\Delta T\end{aligned}\right\}, \tag{32}$$

where the effective compliance ($\bar{\mathbf{S}}_{(\mathrm{c})}$), thermal expansion vectors ($\bar{\boldsymbol{\alpha}}_{(\mathrm{c})}$) and the global effective strains due to release of residual stresses ($\bar{\boldsymbol{\varepsilon}}_{(\mathrm{c})}^{(\mathrm{r})}$) remain to be defined. In eqn (32), the out-of-plane residual stresses are zero ($\boldsymbol{\sigma}_{\mathrm{O}}^{k(\mathrm{r})} = \mathbf{0}$) due to equilibrium and the in-plane residual stresses $\boldsymbol{\sigma}_{\mathrm{I}}^{k(\mathrm{r})}$ are assumed to be known. Furthermore, compatibility and equilibrium conditions for the laminate with micro cracks under the present loading system read

$$\left.\begin{aligned}\boldsymbol{\varepsilon}_{\mathrm{I(c)}}^{k(\mathrm{e})} &= \bar{\boldsymbol{\varepsilon}}_{\mathrm{I(c)}}^{(\mathrm{e})} = \bar{\boldsymbol{\varepsilon}}_{\mathrm{I}}^* \\ \boldsymbol{\sigma}_{\mathrm{O(c)}}^{k(\mathrm{a})} &= \bar{\boldsymbol{\sigma}}_{\mathrm{O(c)}}^{(\mathrm{a})} = \bar{\boldsymbol{\sigma}}_{\mathrm{O}}^*\end{aligned}\right\}. \tag{33}$$

By a substitution of eqns (31), (33) into eqn (32b) and a rearrangement of the resulting equations, the following expressions can be derived:

$$\boldsymbol{\sigma}_{\mathrm{I(c)}}^{k(\mathrm{a})} = (\mathbf{S}_{\mathrm{II}}^k)^{-1}[(\bar{\boldsymbol{\varepsilon}}_{\mathrm{I}}^* - \Delta\boldsymbol{\varepsilon}_{\mathrm{I}}^k) - \mathbf{S}_{\mathrm{IO}}^k\bar{\boldsymbol{\sigma}}_{\mathrm{O}}^* - \boldsymbol{\alpha}_{\mathrm{I}}^k\Delta T] + \boldsymbol{\sigma}_{\mathrm{I}}^{k(\mathrm{r})}, \tag{34a}$$

$$\begin{aligned}\boldsymbol{\varepsilon}_{\mathrm{O(c)}}^{k(\mathrm{e})} &= \boldsymbol{\varepsilon}_{\mathrm{O(c)}}^{k(\mathrm{a})} + \Delta\boldsymbol{\varepsilon}_{\mathrm{O}}^k \\ &= \{(\mathbf{S}_{\mathrm{IO}}^k)^{\mathrm{T}}(\mathbf{S}_{\mathrm{II}}^k)^{-1}(\bar{\boldsymbol{\varepsilon}}_{\mathrm{I}}^* - \Delta\boldsymbol{\varepsilon}_{\mathrm{I}}^k) + [\mathbf{S}_{\mathrm{OO}}^k - (\mathbf{S}_{\mathrm{IO}}^k)^{\mathrm{T}}(\mathbf{S}_{\mathrm{II}}^k)^{-1}\mathbf{S}_{\mathrm{IO}}^k]\bar{\boldsymbol{\sigma}}_{\mathrm{O}}^* \\ &\quad + [\boldsymbol{\alpha}_{\mathrm{O}}^k - (\mathbf{S}_{\mathrm{IO}}^k)^{\mathrm{T}}(\mathbf{S}_{\mathrm{II}}^k)^{-1}\boldsymbol{\alpha}_{\mathrm{I}}^k]\Delta T\} + \Delta\boldsymbol{\varepsilon}_{\mathrm{O}}^k.\end{aligned} \tag{34b}$$

With the help of eqns (16), (19), (26), (30)–(34), the effective compliance tensor ($\bar{\mathbf{S}}_{(\mathrm{c})}$), the thermal expansion coefficients ($\bar{\boldsymbol{\alpha}}_{(\mathrm{c})}$) and the global effective strains due to release of residual stresses ($\bar{\boldsymbol{\varepsilon}}_{(\mathrm{c})}^{(\mathrm{r})}$) can be derived in the same way as was summarized in Section 2.1 for the ordinary laminate theory. The resulting expressions are given below:

$$\bar{\mathbf{S}}_{\mathrm{II(c)}} = \left((\bar{\mathbf{S}}_{\mathrm{II}})^{-1} - \sum_{k=1}^{N} v^k \rho^k (\mathbf{A}^k)^{\mathrm{T}} \sum_{i=1}^{N} \boldsymbol{\beta}^{ki} \mathbf{A}^i \right)^{-1}, \tag{35a}$$

$$\bar{\mathbf{S}}_{\mathrm{IO(c)}} = \bar{\mathbf{S}}_{\mathrm{II(c)}} \left[(\bar{\mathbf{S}}_{\mathrm{II}})^{-1} \bar{\mathbf{S}}_{\mathrm{IO}} + \sum_{k=1}^{N} v^k \rho^k (\mathbf{A}^k)^{\mathrm{T}} \sum_{i=1}^{N} \boldsymbol{\beta}^{ki} \mathbf{B}^i \right], \tag{35b}$$

$$\bar{\mathbf{S}}_{\mathrm{OO(c)}} = (\bar{\mathbf{S}}_{\mathrm{IO(c)}})^{\mathrm{T}} (\bar{\mathbf{S}}_{\mathrm{II(c)}})^{-1} \bar{\mathbf{S}}_{\mathrm{IO(c)}} - (\bar{\mathbf{S}}_{\mathrm{IO}})^{\mathrm{T}} (\bar{\mathbf{S}}_{\mathrm{II}})^{-1} \bar{\mathbf{S}}_{\mathrm{IO}} + \bar{\mathbf{S}}_{\mathrm{OO}} + \sum_{k=1}^{N} v^k \rho^k (\mathbf{B}^k)^{\mathrm{T}} \sum_{i=1}^{N} \boldsymbol{\beta}^{ki} \mathbf{B}^i, \tag{35c}$$

$$\bar{\boldsymbol{\alpha}}_{\mathrm{I(c)}} = \bar{\boldsymbol{\alpha}}_{\mathrm{I}} + \bar{\mathbf{S}}_{\mathrm{II(c)}} \sum_{k=1}^{N} v^k \rho^k (\mathbf{A}^k)^{\mathrm{T}} \sum_{i=1}^{N} \boldsymbol{\beta}^{ki} \mathbf{C}^i, \tag{35d}$$

$$\bar{\boldsymbol{\alpha}}_{\mathrm{O(c)}} = \bar{\boldsymbol{\alpha}}_{\mathrm{O}} + (\bar{\mathbf{S}}_{\mathrm{IO(c)}})^{\mathrm{T}} (\bar{\mathbf{S}}_{\mathrm{II(c)}})^{-1} (\bar{\boldsymbol{\alpha}}_{\mathrm{I(c)}} - \bar{\boldsymbol{\alpha}}_{\mathrm{I}}) + \sum_{k=1}^{N} v^k \rho^k (\mathbf{B}^k)^{\mathrm{T}} \sum_{i=1}^{N} \boldsymbol{\beta}^{ki} \mathbf{C}^i, \tag{35e}$$

$$\bar{\boldsymbol{\varepsilon}}^{(\mathrm{r})}_{\mathrm{I(c)}} = \bar{\mathbf{S}}_{\mathrm{II(c)}} \sum_{k=1}^{N} v^k \rho^k (\mathbf{A}^k)^{\mathrm{T}} \sum_{i=1}^{N} \boldsymbol{\beta}^{ki} \boldsymbol{\tau}^{i(\mathrm{r})}, \tag{35f}$$

$$\bar{\boldsymbol{\varepsilon}}^{(\mathrm{r})}_{\mathrm{O(c)}} = (\bar{\mathbf{S}}_{\mathrm{IO(c)}})^{\mathrm{T}} (\bar{\mathbf{S}}_{\mathrm{II(c)}})^{-1} \bar{\boldsymbol{\varepsilon}}^{(\mathrm{r})}_{\mathrm{I(c)}} + \sum_{k=1}^{N} v^k \rho^k (\mathbf{B}^k)^{\mathrm{T}} \sum_{i=1}^{N} \boldsymbol{\beta}^{ki} \boldsymbol{\tau}^{i(\mathrm{r})}. \tag{35g}$$

In conclusion, eqn (35) defines exact expressions for the thermoelastic properties of composite laminates containing matrix cracks, provided that the $\boldsymbol{\beta}^{ki}$ matrices are known. Under a given loading system, the global effective strains ($\bar{\boldsymbol{\varepsilon}}^{(\mathrm{e})}_{(\mathrm{c})}$) and the global average stresses ($\bar{\boldsymbol{\sigma}}^{(\mathrm{a})}_{(\mathrm{c})}$) for the cracked laminate can be obtained from eqn (32). These results provide the global effective in-plane strains ($\bar{\boldsymbol{\varepsilon}}^{*}_{\mathrm{I}} = \bar{\boldsymbol{\varepsilon}}^{(\mathrm{e})}_{\mathrm{I(c)}}$) and the global average out-of-plane stresses ($\bar{\boldsymbol{\sigma}}^{*}_{\mathrm{O}} = \bar{\boldsymbol{\sigma}}^{(\mathrm{a})}_{\mathrm{O(c)}}$). Finally, eqns (31)–(34) can be applied to recover the ply average stresses and strains. Thus, the laminate theory taking matrix cracks into account is complete. It is observed that the thermoelastic properties of a composite laminate containing micro cracks can be expressed in terms of thermoelastic properties for an uncracked laminate, the micro crack densities and the $\boldsymbol{\beta}^{ki}$ matrices which relate the average crack opening displacements to the crack surface tractions. The determination of these $\boldsymbol{\beta}^{ki}$ matrices will be discussed in the next section.

It should be pointed out that the above equations are valid also for a two-dimensional thin laminate containing matrix cracks. In this case, $\bar{\boldsymbol{\sigma}}_{\mathrm{O}} = \mathbf{0}$ and usually only the in-plane properties are considered.

2.3. *Determination of average crack opening displacements*

The theoretical development above has shown that the thermoelastic properties as well as average local stresses and strains for a micro cracked laminate can be exactly determined provided that the exact solution for the average crack opening displacements or equivalently the $\boldsymbol{\beta}^{ki}$ matrices are known. Exact analytical solutions for the $\boldsymbol{\beta}^{ki}$ matrices are however impossible to derive except for extremely simplified cases. In order to predict thermoelastic properties of micro cracked laminates, approximate solutions for average crack opening displacements must be derived. The quality of the resulting theory therefore strongly depends on the accuracy of the approximate solutions for $\boldsymbol{\beta}^{ki}$.

In the present paper transverse matrix cracks in composite laminates are considered. Thus, the crack surfaces are parallel to the fibre direction in each ply and perpendicular to the laminate plane (the Y_1–Y_3-plane according to the definition of the local coordinate system in Section 2.1). In the case of dilute crack density ($\rho \ll 1$), Gudmundson and Östlund (1992a) showed that the average crack opening displacements in angle plies and cross plies to a surprisingly good accuracy could be determined from the well-known analytical solution of a single crack in an infinite, homogeneous transversely isotropic medium. Thus the average crack opening displacements were independent on the orientation of the neighbouring plies. The average crack opening displacements were generally found to be slightly overestimated by use of the approximate analytical solution. The effect of using approximate average crack opening displacements is however much reduced in application to the determination of thermoelastic properties [see Gudmundson and Östlund (1992a, b, c)].

The fact that the average crack opening displacements for the dilute case showed a very limited dependence on the orientation of the neighbouring plies gives hope for finding closed form but still accurate approximate solutions for nondilute matrix crack densities. If this assumption holds, then the average crack opening displacements for a row of matrix cracks in a ply should be well approximated by the average crack opening displacements for the same row of cracks in an infinite homogeneous transversely isotropic medium which has the same properties as the ply in consideration (see Fig. 3). The stress intensity factors for an infinite row of equidistant cracks in an infinite homogeneous isotropic medium under the action of uniform tractions on crack surfaces are given by Benthem and Koiter (1972) and Tada *et al.* (1973). These stress intensity factor solutions are also valid for the same crack problem in a transversely isotropic medium. The well-known relation between strain energy release rate and stress intensity factors make it possible to determine expressions for average crack opening displacements. Thus, the $\boldsymbol{\beta}^{ki}$ $(k, i = 1, 2, \ldots, N)$ matrices defined in eqn (29) can be approximately determined in this manner. It should be pointed out that the use of these approximate $\boldsymbol{\beta}^{ki}$ $(k, i = 1, 2, \ldots, N)$ matrices in the present model is the single approximation in the determination of thermoelastic properties of composite laminates containing matrix cracks. The use of the analytical solution for an infinite row of equidistant cracks implies of course that the effects of non-equidistant crack spacings cannot be covered by the present model. Experimental observations have however shown that the assumption of equidistant matrix cracks generally is a good representation of the reality [see for example Highsmith and Reifsnider (1982)]. The assumption above implies that there will be no coupling between the crack opening displacements of different plies and that the $\boldsymbol{\beta}^{ki}$ matrices must be diagonal, thus

$$\boldsymbol{\beta}^{ki} = \mathbf{0}, \quad \text{for all } k \neq i, \tag{36a}$$

$$\boldsymbol{\beta}^{kk} = \begin{pmatrix} \beta_1^k & 0 & 0 \\ 0 & \beta_2^k & 0 \\ 0 & 0 & \beta_3^k \end{pmatrix}. \tag{36b}$$

Equation (29) for the average crack opening displacements in ply k can thus be rewritten as

$$\Delta\bar{\mathbf{u}}^k = a^k \boldsymbol{\beta}^{kk} \boldsymbol{\tau}^k \quad (\text{no sum over } k). \tag{37}$$

The diagonal components of the $\boldsymbol{\beta}^{kk}$ matrix can be expressed in closed form as will be shown below. Therefore, the thermoelastic parameters for laminates containing micro

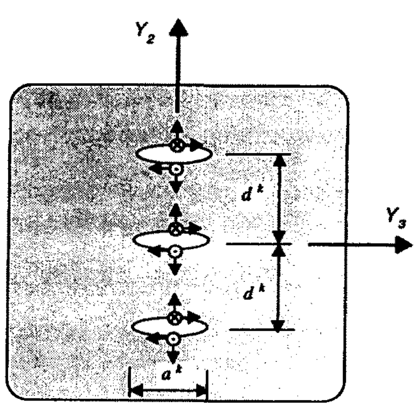

Fig. 3. An infinite row of cracks subjected to crack surface tractions in an infinite transversely isotropic plane.

cracks presented in Section 2.2. can also be written in closed form. The accuracy of the proposed model can only be checked by comparisons to numerical solutions or experimental results. In Section 3, extensive comparisons to numerical finite element results as well as experimental results will be presented.

The stress intensity factors for the crack problem shown in Fig. 3 can be expressed as

$$\left.\begin{aligned} K_{\mathrm{I}} &= f_2\tau_2^k \\ K_{\mathrm{II}} &= f_3\tau_3^k \\ K_{\mathrm{III}} &= f_1\tau_1^k \end{aligned}\right\}, \tag{38}$$

where f_i $(i = 1,2,3)$ can be found in the papers by Benthem and Koiter (1972) and Tada *et al.* (1973). Due to symmetries of the crack problem in Fig. 3, it can be shown that the effective strains in the 1–2-plane for the cracked layer are zero. Similarly, equilibrium enforces the average stress σ_{23} in the cracked layer to vanish. The effective strain and average stress state are thus in exact agreement with the loading conditions for the crack problem in Section 2.2. This is of course the reason for the particular loading system chosen for the crack problem in Section 2.2.

The well-known relation between strain energy release rate and stress intensity factors can now be applied to derive an equation for the average crack opening displacements

$$\tfrac{1}{2}a^k(\Delta\bar{\mathbf{u}}^k)^{\mathrm{T}}\boldsymbol{\tau}^k = \int_0^{a^k} [\gamma_1(f_1\tau_1^k)^2+\gamma_2(f_2\tau_2^k)^2+\gamma_3(f_3\tau_3^k)^2]\,\mathrm{d}a^k, \tag{39}$$

where

$$\left.\begin{aligned} \gamma_1 &= 1/(2G_{\mathrm{TL}}^k) \\ \gamma_2 &= \gamma_3 = (1-\nu_{\mathrm{LT}}^k\nu_{\mathrm{TL}}^k)/E_{\mathrm{T}}^k \end{aligned}\right\}. \tag{40}$$

In eqn (40), E_{T}^k denotes the transverse E-modulus, G_{TL}^k the out-of-plane shear modulus and ν_{TL}^k, ν_{LT}^k the Poisson ratios. Equations (37)–(39) yield relations between the coefficients β_j^k and f_j,

$$\left.\begin{aligned} \beta_1^k &= \frac{2}{(a^k)^2}\gamma_1\int_0^{a^k}(f_1)^2\,\mathrm{d}a^k \\ \beta_2^k &= \frac{2}{(a^k)^2}\gamma_2\int_0^{a^k}(f_2)^2\,\mathrm{d}a^k \\ \beta_3^k &= \frac{2}{(a^k)^2}\gamma_3\int_0^{a^k}(f_3)^2\,\mathrm{d}a^k \end{aligned}\right\}. \tag{41}$$

The first integral in eqn (41) was analytically evaluated. Numerical integration was performed on the other two integrals, and a curve fitting technique with an error less than 0.5% was then employed to generate the resulting expression for β_2^k and β_3^k. Finally, the diagonal components of the $\boldsymbol{\beta}^{kk}$ matrix can be expressed as

$$\left.\begin{aligned} \beta_1^k &= \frac{4}{\pi}\gamma_1\ln[\cosh(\rho^k\pi/2)]/(\rho^k)^2 \\ \beta_2^k &= \frac{\pi}{2}\gamma_2\sum_{j=1}^{10} a_j/(1+\rho^k)^j \\ \beta_3^k &= \frac{\pi}{2}\gamma_3\sum_{j=1}^{9} b_j/(1+\rho^k)^{j-2} \end{aligned}\right\}, \tag{42}$$

where a_j and b_j are given in Table 1.

Table 1. Numerical parameters used in eqns (42), (43)

J	a	b	c
1	0.63666	0.63662	0.25256
2	0.51806	−0.08945	0.27079
3	0.51695	0.15653	−0.49814
4	−1.04897	0.13964	8.62962
5	8.95572	0.16463	−51.24655
6	−33.09444	0.06661	180.96305
7	74.32002	0.54819	−374.29813
8	−103.06411	−1.07983	449.59474
9	73.60337	0.45704	−286.51016
10	−20.34326	—	73.84223

In some cases, surface cracks occur in a laminate. For thin laminate structures, the effects due to surface cracks will become more important and have to be taken into account. In this case, the two-dimensional theory will be employed and only two components of the $\boldsymbol{\beta}^{kk}$ matrix (β_1^k and β_2^k) are required. These two components can be derived in the same manner as above and they can be written as

$$\left.\begin{aligned} \beta_1^{k(s)} &= \frac{8}{\pi}\gamma_1 \ln[\cosh(\rho^k\pi)]/(2\rho^k)^2 \\ \beta_2^{k(s)} &= 2(1.12)^2\left[\frac{\pi}{2}\gamma_2 \sum_{j=1}^{10} c_j/(1+\rho^k)^j\right] \end{aligned}\right\}, \tag{43}$$

where the superscript (s) indicates a surface crack and the parameters c_j are given in Table 1. It is easily shown that

$$\beta_1^{k(s)}(\rho^k) = 2\beta_1^k(2\rho^k), \tag{44a}$$

and that

$$\left.\begin{aligned} \beta_2^{k(s)} &= 2(1.12)^2\beta_2^k \quad \text{as} \quad \rho \to 0 \\ \beta_2^{k(s)} &= \beta_2^k \qquad\qquad\quad \text{as} \quad \rho \to \infty \end{aligned}\right\}. \tag{44b}$$

It is noticed that when ρ^k tends to zero, the coefficients β_j^k approach the dilute results given by Gudmundson and Östlund (1992a). For ρ^k tending to infinity, it can be shown that the results obtained by the present method are in agreement with the results from the infinite limit given by Gudmundson and Östlund (1992b). The present theory is thus in agreement with the asymptotic results for small and infinite matrix crack densities respectively. For intermediate values of ρ^k, the accuracy of the solutions obtained by the present theory has to be checked against numerical or experimental results.

3. RESULTS

In order to verify the efficiency and reliability of the present theory, a number of two- and three-dimensional problems have been studied by the present theory and compared either to finite element calculations or to experimental results presented in the literature. Two kinds of laminate systems have been considered, thin cross-ply laminates with micro cracks in one type of ply and angle-ply laminates with micro cracks in both types of plies. In the finite element calculations, periodic cells with appropriate periodic boundary conditions were employed and the FE program ABAQUS was used. A detailed description of the finite element modelling was given by Gudmundson and Östlund (1992c). The ply material properties for the laminates which were used in the verifications are presented in Table 2. Some typical results for cross-ply and angle-ply composite laminates will be presented in the following sections.

Table 2. Material properties for GFRP and CFRP unidirectional plies

Type	E_L (GPa)	E_T (GPa)	ν_{LT}	ν_{TT}	G_L (GPa)	Ply thickness	α_L $10^{-6}\,^{\circ}C^{-1}$	α_T $10^{-6}\,^{\circ}C^{-1}$
GFRP	41.7	13	0.3	0.42	3.4	0.203	6.72	29.3
CFRP	142	9.85	0.3	—	4.48	0.127	—	—

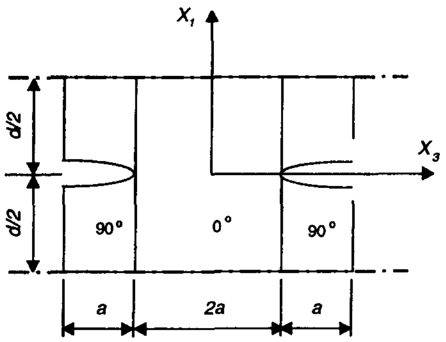

Fig. 4. A representative periodic cell for a cross-ply laminate with surface cracks.

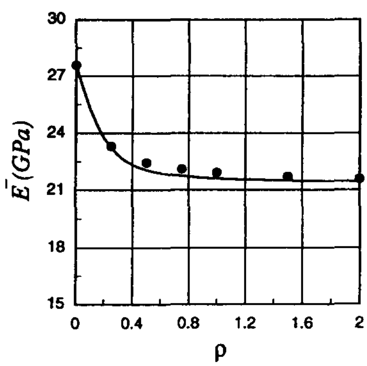

Fig. 5(a). $\bar{E}$-modulus as a function of micro crack density in the 90° plies for the cross-ply laminate with surface cracks (see Fig. 4). The solid line denotes the results by the present method and the symbol the results from finite element calculations.

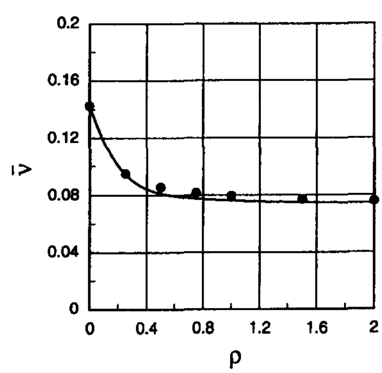

Fig. 5(b). Poisson's ratio as a function of micro crack density in the 90° plies for the cross-ply laminate with surface cracks (see Fig. 4). The solid line denotes the results by the present method and the symbol the results from finite element calculations.

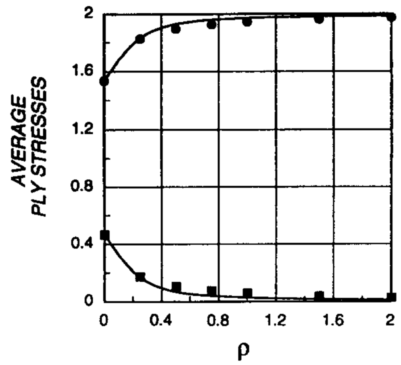

Fig. 6. Average ply stresses (in global coordinate system) as functions of micro crack density in the 90° plies for the cross-ply laminate with surface cracks (see Fig. 4). The loading is a global average stress $\bar{\sigma}_{11} = 1.0$ MPa. The solid lines denote the results by the present method and the symbols the results from finite element calculations, where ● = σ_{11} for the 0° ply and ■ = σ_{11} for the 90° plies.

3.1. *Cross-ply laminate*

Comparisons to results obtained either by finite element calculations or by experimental studies are here presented for thin cross-ply laminates. Transverse matrix cracks are assumed to exist only in the 90° plies and to cover the whole width of the plies.

A thin cross-ply GFRP laminate with layup $[90/0]_s$ is first considered. The geometry of a representative periodic cell with surface cracks in both 90° plies is shown in Fig. 4. In Figs 5(a, b), the effective E-modulus and Poisson's ratio as functions of micro crack densities obtained by the present method are compared to finite element calculations. It is observed that the results generated by the present method agree very well with the finite element calculations for both small and large micro crack densities. Relatively larger differences exist for intermediate matrix crack densities. However, the maximum error in this example is only about 5%. In addition, average ply stresses in the cracked laminate resulting from a unidirectional loading of $\bar{\sigma}_{11} = 1.0$ MPa has also been studied. In Fig. 6, the average ply stresses are compared to finite element calculations at discrete matrix crack densities. It is again observed that the agreement is good for all crack densities.

The normalized E-modulus for some $[0/90]_s$ and $[0_2/90_2]_s$ CFRP laminates as functions of micro crack densities is presented in Fig. 7. The experimental data shown in Fig. 7 are based on the results given by Groves *et al.* (1987). A good agreement between experimental and theoretical results can be observed. In Fig. 8, the normalized E-moduli for $[0/90_3]_s$ GFRP laminates obtained by the present theory are compared to the experimental results given by Highsmith and Reifsnider (1982). It is observed that the differences are relatively larger for larger matrix crack densities. The reason for this discrepancy could be that

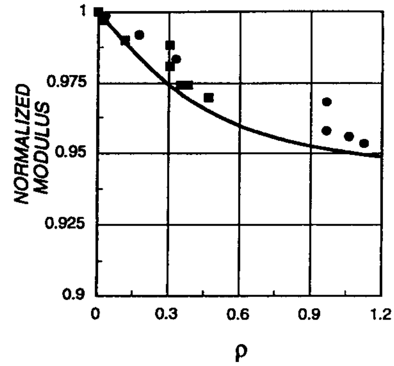

Fig. 7. Normalized E-modulus as a function of microcrack density in the 90° plies for the CFRP cross-ply laminates. The solid line denotes the prediction by the present theory and the symbols the experimental results by Groves *et al.* (1987), where ■ represents the results for $[0/90]_s$ laminates and ● for $[0_2/90_2]_s$ laminates.

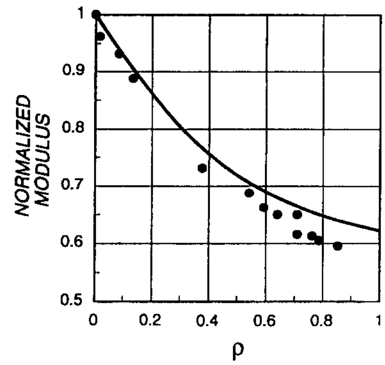

Fig. 8. Normalized E-modulus as a function of micro crack density in the 90° plies for the GFRP $[0/90_3]_s$ laminate. The solid line denotes the prediction by the present theory and the symbol the experimental results by Highsmith and Reifsnider (1982).

additional damage modes which are not included in the present theory are present in the experimental studies. However, the differences in Fig. 8 are not so large. The predictions by the present theory are still quite good.

3.2. *Thick angle ply laminate*

In this section, some thick angle-ply GFRP laminates with matrix cracks in both plies and covering the whole thickness of the laminate are investigated. In order to study the accuracy of the present method at varying layup configurations, laminates with layup $[\pm 55]_N$ and $[\pm 67.5]_N$ have been considered. Here N is assumed to be large so that three-dimensional theory can be applied. Since a laminate with layup $[\pm 45]_N$ really is a cross-ply laminate, this kind of layup is thus not included here. The finite element method has been employed to verify the efficiency and reliability of the present theory. Due to symmetry only one half of the periodic cell has been modelled by finite elements. The periodic cell geometry and the finite element mesh are illustrated in Fig. 9. In Fig. 9, the coordinate system (X_1, X_2, X_3)

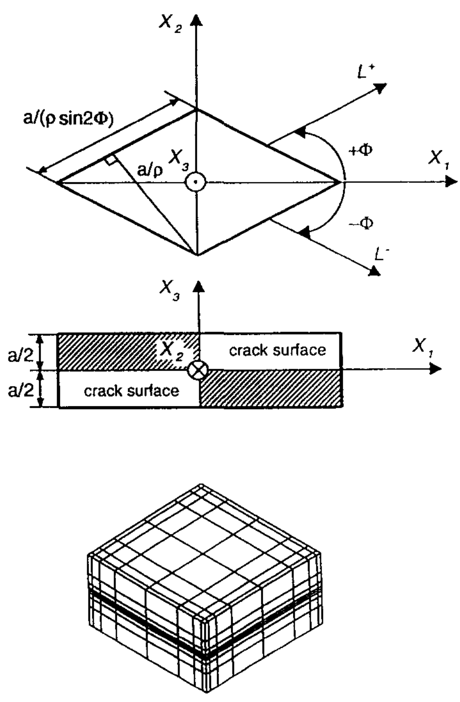

Fig. 9. Coordinate systems, geometry and finite element mesh of a representative periodic cell for a thick angle ply laminate with micro cracks in both types of plies.

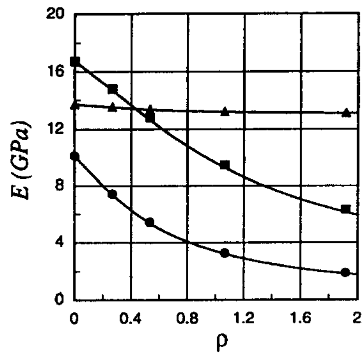

Fig. 10(a). E-moduli as functions of micro crack density for the angle ply ($\phi = \pm 55°$) laminate. The solid lines denote the results by the present theory, the symbols the results from finite element calculations, where ● = $\bar{E}_1$, ■ = $\bar{E}_2$ and ▲ = $\bar{E}_3$.

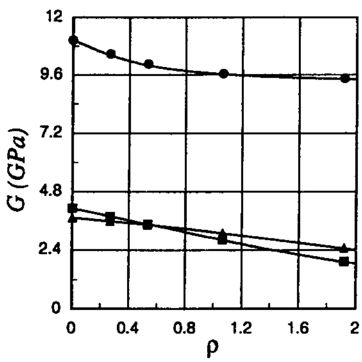

Fig. 10(b). Shear moduli as functions of micro crack density for the angle ply ($\phi = \pm 55°$) laminate. The solid lines denote the results by the present theory, the symbols the results from finite element calculations, where ● = $\bar{G}_{12}$, ■ = $\bar{G}_{13}$ and ▲ = $\bar{G}_{23}$.

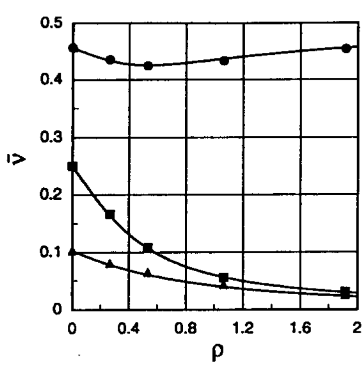

Fig. 10(c). Poisson ratios as functions of micro crack density for the angle ply ($\phi = \pm 55°$) laminate. The solid lines denote the results by the present theory, the symbols the results from finite element calculations, where ● = $\bar{\nu}_{12}$, ■ = $\bar{\nu}_{13}$ and ▲ = $\bar{\nu}_{23}$.

represents the global coordinate system. The directions L^+ and L^- are aligned to the fibre directions in the corresponding ply. The local coordinate systems are not shown in Fig. 9.

Effective global engineering constants and thermal expansion coefficients as functions of matrix crack densities are presented in Figs 10(a–d) for the $[\pm 55]_N$ laminate and in Fig. 11(a–d) for the $[\pm 67.5]_N$ laminate. In addition, the prediction of average ply stresses by the present theory has been studied for a unidirectional loading case, $\bar{\sigma}_{11} = 1.0$ MPa. Comparisons of average ply stresses to finite element calculations are presented in Figs 12(a, b). It is observed that the agreement is generally quite good for all cases. It should be pointed out that each solid line in Figs 10–12 consists of more than 150 data points. In order to illustrate the efficiency of the present method, the CPU time used by the present

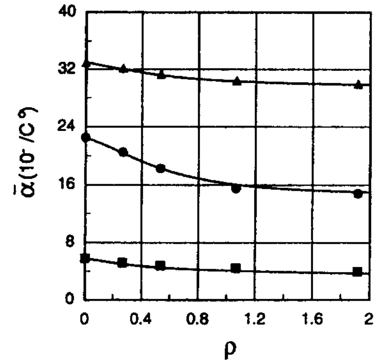

Fig. 10(d). Thermal expansion coefficients as functions of micro crack density for the angle ply ($\phi = \pm 55°$) laminate. The solid lines denote the results by the present theory, the symbols the results from finite element calculations, where ● = $\bar{\alpha}_{11}$, ■ = $\bar{\alpha}_{22}$ and ▲ = $\bar{\alpha}_{33}$.

method and by finite element calculations may be compared. It took a Macintosh SE/30 computer about 5 min to generate all theoretical results shown in Figs 10–12 in comparison to about 4 CPU hours for the finite element calculation of one single layup and one particular crack density on a DEC 3100 Work Station.

4. DISCUSSION

A model for the thermoelastic properties of composite laminates containing plies with transverse matrix cracks has been developed. The model can handle laminates of arbitrary

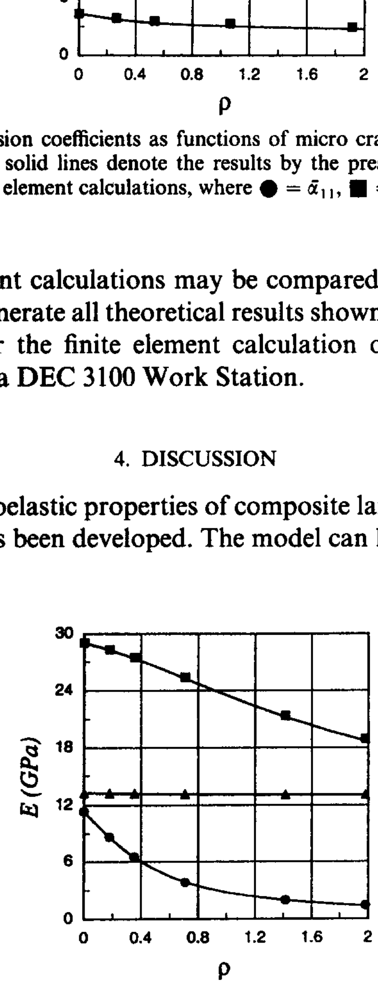

Fig. 11(a). E-moduli as functions of micro crack density for the angle ply ($\phi = \pm 67.5°$) laminate. The solid lines denote the results by the present theory, the symbols the results from finite element calculations, where ● = $\bar{E}_1$, ■ = $\bar{E}_2$ and ▲ = $\bar{E}_3$.

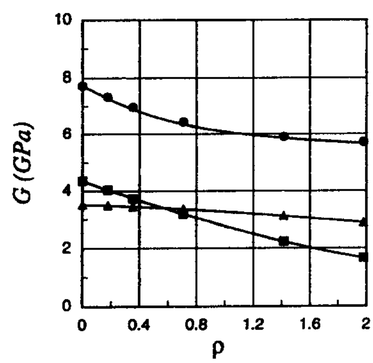

Fig. 11(b). Shear moduli as functions of micro crack density for the angle ply ($\phi = \pm 67.5°$) laminate. The solid lines denote the results by the present theory, the symbols the results from finite element calculations, where ● = $\bar{G}_{12}$, ■ = $\bar{G}_{13}$ and ▲ = $\bar{G}_{23}$.

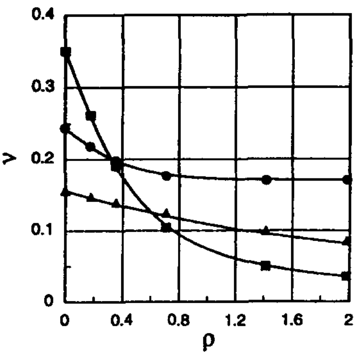

Fig. 11(c). Poisson ratios as functions of micro crack density for the angle ply ($\phi = \pm 67.5°$) laminate. The solid lines denote the results by the present theory, the symbols the results from finite element calculations, where ● = $\bar{\nu}_{12}$, ■ = $\bar{\nu}_{13}$ and ▲ = $\bar{\nu}_{23}$.

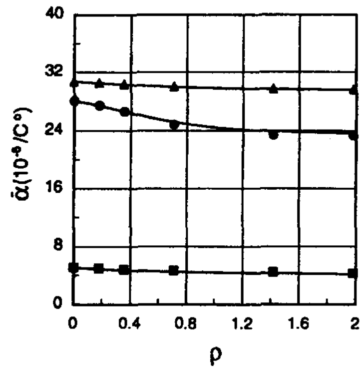

Fig. 11(d). Thermal expansion coefficients as functions of micro crack density for the angle ply ($\phi = \pm 67.5°$) laminate. The solid lines denote the results by the present theory, the symbols the results from finite element calculations, where ● = $\bar{\alpha}_{11}$, ■ = $\bar{\alpha}_{22}$ and ▲ = $\bar{\alpha}_{33}$.

layup configurations and there is no limitation on possible matrix crack densities which can be treated. The fact that the model is formulated in closed form analytical expressions is another nice feature. The model is only based on known parameters such as ply property data. In comparison to alternative models for the prediction of stress–strain relationships of matrix cracked laminates, the present model has some clear advantages. First of all, the present model is more versatile than other models, since there is no restriction concerning laminate layup nor micro crack densities. Alternative models such as the shear lag theory or the Hashin (1985, 1987, 1988) model have generally only been developed for cross-ply

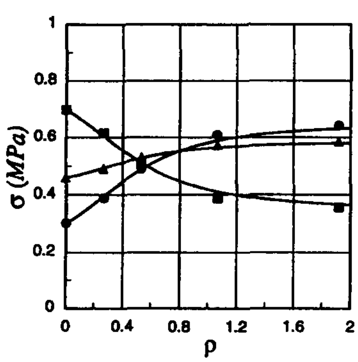

Fig. 12(a). Average ply stresses (in the local coordinate system) for the −55° ply as functions of micro crack densities. The loading is a global average stress of $\bar{\sigma}_{11}$ = 1.0 MPa. The solid lines denote the results by the present theory and the symbols the results from finite element calculations where ● = σ_{11}, ■ = σ_{22} and ▲ = σ_{12}.

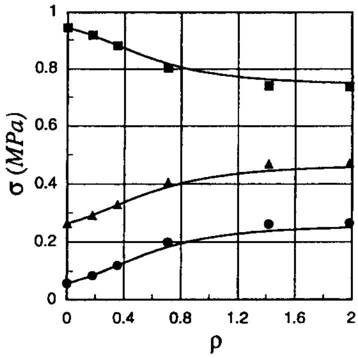

Fig. 12(b). Average ply stresses (in local coordinate system) for the $-67.5°$ ply as functions of micro crack density. The loading is a global average stress of $\bar{\sigma}_{11} = 1.0$ MPa. The solid lines denote the results by the present theory and the symbols the results from finite element calculations where ● $= \sigma_{11}$, ■ $= \sigma_{22}$ and ▲ $= \sigma_{12}$.

laminates. Studies on layup configurations other than cross plies are very scarce in the literature. Secondly, the accuracy of the present model is at least as good as alternative models for cross-ply laminates. In addition, the analytical formulation of the present model makes it very easy to implement on a computer. Hence, in the authors' opinion the model presented in this paper is generally applicable and accurate enough for simulations of matrix cracked laminates.

The present paper has only considered the stress–strain relationship at given matrix crack densities. In order to simulate the behaviour of a mechanically loaded structure the model presented here must be complemented by criteria for matrix crack initiation and growth. A crack initiation and growth model could be expressed in terms of ply stresses/strains or energy release rates. The ability to predict average ply stresses and strains has already been demonstrated in the paper. The energy release rate for the creation of a new matrix crack surface area can be expressed in terms of the derivatives of stiffnesses or compliances with respect to matrix crack densities. Since the compliances as functions of micro crack densities are known from the present model, energy release rates can be accurately determined. In future work it is planned to include criteria for matrix crack initiation and growth in the present model.

An effect which has not been treated by the present model is eventual crack closures of matrix cracks. It would be possible to include this effect within the model, but it would be quite complicated because of the crack closure induced nonlinearities. This improvement of the model is therefore left for future developments.

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